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ABSTRACT

This study was presented at the 1979 meeting of the American Education Research Association (AERA), San Francisco. It investigated the relationship between the structure of simple addition and subtraction problems and the types of processes children use to solve them prior to receiving formal arithmetic instruction. Specific strategies which young children use to carry out addition and subtraction computations were also studied. Forty-three first-grade children were individually administered 20 problems representing 10 different problem types. Results showed that before receiving instruction, children: (1) can successfully solve addition and subtraction problems; (2) use solution processes which model the structural problem type; and (3) rely on a variety of counting strategies for computation. (Author/MM)

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THE EFFECT OF PROBLEM STRUCTURE ON FIRST-GRADERS' INITIAL SOLUTION PROCESSES FOR SIMPLE ADDITION AND SUBTRACTION PROBLEMS

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A major goal of mathematics instruction is to teach children to apply their mathematical skills to solve problems. It is frequently assumed that children must first master computational skills before they can begin to apply them to the solution of problems. However, it has been clearly demonstrated that children develop a variety of informal strategies for solving mathematical problems independent of instruction (c.f. Ginsburg, 1977; Resnick, Note 1). In fact many of the informal strategies are more sophisticated and demonstrate more insight than the formal procedures that are a part of instruction. This raises the hypothesis that, rather than depending on a prior knowledge of computational skills, simple problems may give meaning to basic mathematics operations.

The focus of this study is on children's initial concepts of addition and subtraction as reflected in their ability to solve verbal and concrete problems representing addition and subtraction operations. The working hypothesis of the study is that prior to formal instruction many children can solve a variety of different problems involving addition and subtraction operations. By identifying the processes that children use to solve different problems, the study attempts to gain a clearer picture of children's initial concepts of addition and subtraction as well as to provide some insights into their problem solving abilities.

An initial concern of this study was to characterize basic problem types that provide different interpretations of addition and subtraction operations. We have identified four semantically different

classes of problems that represent addition and subtraction: Joining/Separating, Part-Part-Whole, Comparison, and Equalizing.

For problems in the Joining/Separating class there is an initial quantity and some direct or implied action that causes a change in the quantity. For problems in the Part-Part-Whole class there is no action direct or implied. This class represents situations in which there are two quantities which may be considered individually or as part of a whole. As the name implies problems in the Comparison class involve the comparison of two quantities. This includes problems in which the difference between two given quantities is to be found and problems in which one of two quantities and the magnitude of the difference between them is given and the second quantity is the unknown. Equalizing problems share characteristics of both Joining/Separating and Comparison problems. There is implied action on a given set, but a comparison is also involved.

The relationship between these four classes of addition and subtraction problems is illustrated in the 2 x 2 matrix in Figure 1.

Insert Figure 1 about here

There are two dimensions on which the four classes of problems differ. One major distinction is based on whether the problems describe action or static relationships. In Joining-Separating and Equalizing problems there is direct or implied action in which one set is joined to or separated from another set. On the other hand, both Part-Part-Whole and

Comparison problems involve relationships between quantities, and there is no action implied or direct. The second major distinction is based on set inclusion relationships. In both the Joining/Separating and Part-Part-Whole classes two of the entities involved in the problem are necessarily a subset of the third. In other words, either the unknown quantity is made up of the two given quantities or one of the given quantities is made up of the other given quantity and the unknown. For Comparison and Equalizing problems this is not the case.

By varying the unknown quantity or the nature of the action in the problems (joining or separating), both addition and subtraction operations can be represented by problems in each of the four classes (Tables 1 and 2).

Insert Table 1 and Table 2 about here

Problem structure as defined by the four basic classes of problems described above was one major variable included in the study. The second was mode of representation. Each problem type was presented physically using sets of concrete objects (Table 2) and through a verbal problem describing the action or relationship (Table 1). In problems involving concrete objects, children can operate directly on the sets of concrete objects. Whereas in the verbal problems, children who depend on concrete methods of solution must first represent the quantities in the problem using concrete representations like fingers or cubes.

There are other dimensions upon which addition and subtraction problems differ that may significantly affect children's performance. These include: the specific numbers in the problem, syntax, vocabulary level, and the number of words in a problem. These factors were controlled but not systematically investigated as part of this study.

One primary object of the study was to identify how successful children are at solving different types of addition and subtraction problems prior to formal instruction in these operations. In other words, one purpose of this study was to determine whether children can independently generate solutions to certain addition and subtraction proolems and identify which types of problems are most difficult for them to solve. This information should provide some basis for deciding which types of problems children readily understand as initial models of addition and subtraction.

The second major objective of the study was to characterize the processes or strategies that children use to solve different problems and identify the factors that lead to the selection of different strategies. There are a number of general patterns that children' solution strategies may follow. One hypothesis is that children develop single strategies for addition and subtraction and use them in all appropriate problems. For example, an individual child might use a separating strategy to solve all subtraction problems. A competing hypothesis is that children' strategies match a given problem's structure and they model the implied actions or relationships in the problem. Different

strategies imply different conceptions of addition and subtraction, and identifying the processes that children use to solve different problems should provide some insight into their understanding of addition and subtraction operations.

Background

Other attempts to characterize fundamentally different classes of addition and subtraction problems are generally consistent with the above analysis. Greeno (Note 2) and Nesher and Katriel (Note 3) identify three distinct classes of problems that they propose are necessary and sufficient to characterize all problems that can be solved by a single operation of addition and subtraction. These correspond to the Joining/Separating, Part-Part-Whole, and Comparison classes. On the other hand, Steffe (1970) and LeBlanc (Note 4) only differentiate between problems in terms of action and no action.

Although there have been a number of studies involving the solution of simple verbal problems, there is very little research or analysis of the processes that children use to solve simple addition or subtraction problems. Greeno (Note 2) has hypothesized that certain types of problems are associated directly with addition or subtraction operations. Others are first transformed to one of the representations that is directly associated with an operation. In general the canonical forms (problems that are naturally represented as $a + b = \square$ or $a - b = \square$) are directly

translated to addition or subtraction operations while non-canonical forms (e.g. missing addend problems) are first transformed to Part-Part-Whole representations. Greeno has little empirical support for this analysis. Although there is some support for the theory in Case's (1978) work, the analysis is not consistent with the results of a study by Riley (Note 5).

Riley's study does demonstrate, however, that the problem structure significantly affects problem difficulty. She found that all forms of Joining/Separating problems were relatively easy, the subtraction form of the Part-Part-Whole problems were somewhat more difficult, and the Comparison problems were still more difficult especially when the comparison amount or reference amount was unknown. These results support the findings of Nesher and Katriel (Note 3) that static subtraction problems are significantly more difficult than parallel dynamic problems.

A number of studies have investigated the strategies that children use to solve open addition and subtraction sentences. The largest collection of such studies have relied upon a paradigm that attempts to match response latencies for subjects solving a variety of problems of a given type and the regression equations of possible solution strategies. For addition, three basic strategies have been identified (Groen and Parkman, 1972; Suppes and Groen, 1967). To calculate the answer to 3 + 5 = ?, the most basic strategy involves counting to 3 and then counting on 5 more times. A somewhat more sophisticated and efficient strategy is to start counting at the first number. In this case it would mean starting at 3 and counting on 5 more times. The most sophisticated and efficient

strategy is to start counting at the larger of the two numbers. In the above problem this would mean starting at 5 and counting on 3 more times. For sums less than 10, this last strategy provides the best model of first-graders' responses (Groen and Parkman, 1972).

As part of a similar analysis of subtraction, two basic strategies were hypothesized (Woods, Resnick, and Groen, 1975). To solve 9 - 6 = ?, children might count down 6 units from 9, or they might count up from 6 until they reach 9 and keep track of the number of units counted. For this particular problem the second strategy would require fewer steps, while the counting down strategy would be more efficient for 8 - 2 = ?. The results of this study indicate that by the second grade four-fifths of the children used a choice strategy by which they choose the most efficient of the two strategies and by the fourth grade the responses of all children best fit a model predicted by such a strategy (Groen and Parkman, 1972).

These data indicate that as children mature they develop more sophisticated and efficient counting strategies. Furthermore, the results of another study indicate that these strategies are developed independent of instruction; and the strategies that children construct for themselves are frequently more sophisticated and efficient than the ones they are taught (Groen and Resnick, 1977).

Method

Tasks

Two different problems were selected from each of the Part-Part-Whole, Comparison, and Equalizing classes and four from the Joining/Separating class. One addition and one subtraction problem were selected from the Part-Part-Whole and Comparison classes. Since Equalizing problems most naturally represent subtraction operations, two subtraction problems were selected from this class. One involved increasing the smaller quantity, and the other involved decreasing the greater quantity. Two distinct types of action are included in the Joining/Separating class, joining and separating. Since these are the most commonly used problems in elementary school mathematics programs, more than two problems were needed to adequately represent this class of problems. The final decision was to include one joining addition problem, two joining missing addend problems, and one separating problem. Two missing addend problems were included because two distinct forms of the missing addend problem were identified, and it was not clear which most adequately represents this type of problem. The structural difference between the two problems is most clearly illustrated for problems presented with concrete materials (see Table 2).

For each of the 10 types of problems, a verbal problem and a problem involving action or relationships between sets of cubes were generated. The verbal problems are presented in Table 1 and the concrete problems

are listed in Table 2. The concrete problems were designed to model the action in the corresponding verbal problems as closely as possible. All problems were constructed so as to provide a relatively simple example of the given type with respect to syntax, vocabulary, sentence length, familiarity of problem situations, etc.

The number triples for the problems were selected to conform to the following specifications: (a) Each of the addends was greater than 2 and less than 10, (b) their sum was greater than 10 and less than 17, and (c) the absolute value of the difference between the two addends was greater than 1. These rules generated the following set of 10 triples: (3,8,11), (3,9,12), (4,7,11), (4,8,12), (4,9,13), (5,7,12), (5,8,13), (5,9,14), (6,8,14), (6,9,15). This number domain was selected because the numbers were small enough so that the problems could be reasonably modeled using concrete objects but were large enough so that it was unlikely that many children would have already learned the addition or subtraction combinations. It was also more likely that the children's strategies would be observable with numbers of this size than with smaller numbers. Doubles and near doubles were eliminated because it was hypothesized that children may operate differently with those combinations (cf. Groen and Parkman, 1972).

The number triples were equally distributed over the set of problems so that each number triple was paired with each problem either four or five times. Each subject received each number triple exactly once within the set of verbal problems and once within the set of concrete problems,

but different subjects received different combinations for a given problem. The number pairings for each subject were made so that the verbal problems contained the same number combination as the corresponding concrete problem. For the addition problems, the smaller of the two addends was always presented first. For the subtraction problems, the larger of the two addends was always selected for the unknown.

Subjects

The subjects for the study consisted of the 43 children in the two first-grade classes of a parochial schools that draws students from a predominantly middle class area of Madison, Wisconsin. Mathematics instruction in both classes consisted of topics 15 to 22 of the Developing Mathematical Processes (DMP) program (Romberg, Harvey, Moser, and Montgomery, 1974). At the time of testing in early February, only two arithmetic topics had been covered, Writing Numbers and Comparison Sentences. The other six topics deal with measurement and geometry. The topic of Comparison Sentences introduced the notion of a mathematical sentence, though it only deals with representing a static relation (equality) between two numbers. Thus, at the time the children were tested, no formal instruction in symbolic representation of addition and subtraction had been given. On the other hand, several lessons including problem situations involving joining, separating, part-part-whole and comparison had been presented. In those instances, modeling with objects to determine the solutions had been suggested.

Procedures

This study relied upon individual interviews with children to identify the processes that they were using to solve each of the problems. Ginsburg (1976) has made a strong case that this type of clinical technique is the most appropriate method for assessing children's mathematical behavior. Each problem was individually administered to each subject by one of two experimenters.

For the concrete problems the appropriate sets were constructed by the experimenter using Unifix cubes of two colors. Subjects were instructed to count the sets to determine the number of elements in each set. If subjects made a counting error, they were instructed to check their result. After subjects had determined the number of elements in the sets, the action or relationship specified by the problem was described by the experimenter; and subjects were asked to solve the given problem. Extra cubes were available if subjects needed them to solve the problem.

The verbal problems were read to the subjects by the experimenter. Problems were reread as often as necessary so that remembering the given numbers or relationships was not a factor. A set of cubes identical to those used in the concrete problems were made available to the subjects. They were encouraged to solve the problem without the cubes but were told to use the cubes if they needed them or were not sure of their answer. There was not strong pressure either to use the cubes or to solve the problems without them, but if subjects were floundering they were reminded that they could use cubes to find the answer.

If a solution process that a subject used was obvious, the experimenter coded the response and went on to the next problem. If it was not completely clear how a subject had found a given answer, the subject was asked to describe how the answer was found. The experimenter continued questioning until it was clear what strategy the subject had used or it was clear that no clear explanation was forthcoming.

The testing required two sessions that lasted 10 to 15 minutes each. Half the subjects received the 10 concrete problems in the first session and the 10 verbal problems in the second session. For the other half of the subjects this order of administration was reversed. Subjects were randomly assigned to one of these administration conditions.

The order of the tasks within the concrete and verbal groups was also randomized for each subject. Thus, each subject received a different sequence of problems, but each subject received the concrete problems in the same order that they received the verbal problems.

Results

Addition

Addition responses were coded in terms of (a) the mode of representation used to generate the solution, (b) the solution strategy, (c) correct or incorrect, and (d) when appropriate, the type of error.

Mode of representation. Two basic modes of physical representation were used by the subjects to model addition problems, and a number of

subjects did not use any physical representation to solve a given problem. Since paper and pencil were not available, symbolic or pictorial representations were not possible. Responses were coded as follows:

(C) Cubes - Cubes were used to represent the action or relationships in the problem.

(F) Fingers - A child used fingers rather than cubes to represent the action or relations in the problem situation or as a tracking device to remember the numbers in some counting sequence.

(N) No physical representation - There was no observable use of cubes or fingers.

Strategy. The three basic counting models identified by Parkman and Groen (1971) were also found in this study. Several strategies that were not based on counting were also identified.

(CA) Counting all - If cubes or fingers were used as models or when counting was done in the child's head, the counting sequence started with one and ended with the number representing the total of the two given sets. In other words the child counted the complete union of the sets represented in the problem.

(CF) Counting on from first number - When counting cubes, fingers, or mentally, the counting sequence began either with the first (smaller) given number in the problem or the successor of that number.

(CL) Counting on from larger number - The counting sequence began with the larger (second) given number or with the successor of that number.

(KF) Know fact - The child gave an answer with the justification that it was the result of knowing some basic addition fact.

(H) Heuristic - Heuristic strategies were employed by a few children to generate solutions from a small set of known basic facts. These strategies usually were based on doubles or numbers whose sum was 10. For example, to solve a problem representing $6 + 8 = ?$ a subject responded that $6 + 6 = 12$ and $6 + 8$ was just 2 more than 12. In another example involving $4 + 7 = ?$ a subject responded that $4 + 6 = 10$ and $4 + 7$ was just 1 more than 10.

(U) Uncodable - A correct answer was provided but the interviewer was unable to determine what strategy a child was employing.

Correct-Incorrect. Responses were coded correct or incorrect based upon whether an appropriate strategy was used and whether the strategy was applied without error to get a correct answer.

(V) A valid or correct strategy was used.

(A) The correct answer was found.

Errors. Two types of errors were identified.

(CE) The child used a correct strategy but misapplied it by miscounting or perhaps forgetting one of the given numbers and thereby found an incorrect answer for the solution.

(E) This category includes use of an incorrect or inappropriate strategy, an unidentifiable strategy, an incorrect guess, or failure to generate an answer of any kind.

The results of the six addition problems are summarized in Table 3.

Insert Table 3 about here

The most popular form of representation was C. This is not surprising given the fact that with the number domain used in this study it is difficult to model the operations using fingers or to count in one's head. Different strategies tended to be paired with different modes of representation. Almost all students who used cubes used a counting all strategy (CA). In fact, to solve the problem that represented 3 + 8 = ?, subjects would generally construct a set of 3 cubes, then a set of 8 cubes, and then count the number of cubes in the union of the two sets. They did not even take advantage of the fact that they had already counted both the set of 3 and the set of 8 and did not need to recount them. In fact in counting the union of the two sets, many subjects were very careful to count one set first and then the other. If a subject constructed both sets but did not recount them both, the response was coded as the appropriate counting on strategy. There were only 4 such responses in all three verbal problems and a total of 14 in the three concrete problems.

Counting on from the first number given in the problem (CF) and counting on from the larger number (CL) were the dominant counting strategy for subjects who used fingers (F) or no physical model (N). Only 3 subjects who used fingers in any of the six problems used the counting all (CA) strategy, and only one who used no model gave a counting all explanation. Again this is not especially surprising since the counting all

process is rather difficult to keep track of in one's head or on fingers. Furthermore, it is not unreasonable to conjecture that the ability to deal with numbers without concrete referents is related to the ability to use the more abstract counting on strategies.

Overall subjects were extremely successful in solving both the joining and part-part-whole addition problems. For each problem more than 88% of the subjects used a correct strategy (V), and over 80% found the correct answer (A). The Comparison problems turned out to be much more difficult. Subjects seemed to have a difficult time understanding the problem. In the verbal problem, 23 subjects gave one of the given numbers as their response. They did not seem to be able to understand that "Jeff had 5 more pieces of gum than Ralph" and interpreted it as "Jeff had 5 pieces of gum." Children could deal with the "more than" relation in the subtraction Comparison problem and the two Equalizing problems. It seems to be this particular context that gave them difficulty. It may be that for children of this age "more" implies a comparison of two sets, and they cannot understand it in terms of incrementing a given set.

The patterns of solution were almost identical for the Joining and Part-Part-Whole Problems. A comparison of the verbal and concrete problems indicates that subjects used cubes and a counting all strategy to a somewhat greater extent in the concrete Joining and Part-Part-Whole problems. In contrast the two Comparison problems were not so similar. Ten more subjects correctly solved the concrete problems than solved the verbal problem. Furthermore only 3 subjects gave one of the given numbers as their response to the concrete problem as opposed to 23 for the verbal problem.

Subtraction

The same mode of representation and correct-incorrect criteria were used for subtraction as were used for addition. But subtraction involves different strategies than addition, and several errors were identified that were sufficiently prevalent in subtraction problems to warrant classification.

Strategies. Four basic subtraction responses were identified. They take on a different form depending upon the model chosen. For concrete representations they are:

(S) Separating - The child models the larger given set and then takes away or separates, one at a time, a number of cubes equal to the given number in the problem. Counting the set of remaining cubes yields the answer.

(ST) Separating to - After the larger set is modeled, the child removes cubes one at a time until the remainder is equal to the second given number of the problem. Counting the number of cubes removed gives the answer.

(AO) Adding on - The child sets out a number of cubes equal to the smaller given number (an addend). The child then adds cubes to that set one at a time until the new collection is equal to the larger given number. Counting the number of cubes added on gives the answer.

(M) Matching - The child puts out two sets of cubes, each set standing for one of the given numbers. The sets are then matched one-to-one. Counting the unmatched cubes gives the answer.

Three more abstract counting strategies were also observed. These are the analogues to the first three concrete strategies listed above.

(CB) Counting back - A child initiates a backwards counting sequence beginning with the given larger number. The backwards counting sequence contains as many counting number words as the given smaller number. The last number uttered in the counting sequence is the answer. This is the counting analogue to the separating (S) strategy.

(CT) Counting back to - A child initiates a backwards counting sequence beginning with the larger given number. The sequence ends with the smaller number. By keeping track of the number of counting words uttered in this sequence, either mentally or by using fingers or perhaps cubes, the child determines the answer to be the number of counting words used in the sequence. This is the counting analogue to the Separating to (ST) strategy.

(CU) Counting up from smaller - A child initiates a forwards counting sequence beginning with the smaller given number. The sequence ends with the larger given number. Again, by keeping track of the number of counting words uttered in the sequence, the child determines the answer. This is the counting analogue to the Adding on (AO) strategy.

The known fact (KF), heuristic (H), and uncodable (U) categories follow the same rules as the corresponding addition categories.

Errors. The (CE) and (E) error categories were coded on the same basis as they were for addition. Two other errors occured with sufficient frequency to be of interest.

(G) Given number - A child simply responds that the answer is one of the two numbers given in the original problem.

(O) Wrong operation - A child uses an addition strategy or the given answer strongly indicates that addition or an addition basic fact was used.

Certain of the strategies naturally model the action described in specific problems. The Separating problem is most clearly modeled by the separating (S) strategy or the related counting back (CB) strategy. On the other hand, the implied joining action of the Joining (missing addend) problems is most closely modeled by the adding on (AO) and counting up (CU) strategies. Comparison problems, on the other hand, deal with relationships between sets rather than action. In this case the matching strategy (M) appears to provide the best model.

For the Part-Part-Whole and qualizing problems the situation is more ambiguous. In the Part-Part-Whole problems there is no implied action so neither the separating or adding on strategies seem more appropriate. But since one of the given quantities is a subset of the other, there are not two distinct sets that can be put into one-to-one correspondence.

For the equalizing problems the situation is reversed. Since the equalizing problems involve both a comparison and some implied action, two different strategies might be seen as appropriate. The addition equalizing problems involve a comparison of two quantities and a decision of how much should be joined to the smaller quantity to make them

equivalent. Thus, both the matching (M) or the adding on (AO) and (CU) strategies might be appropriate. For the subtracting Equalizing problems the implied action involves removing elements from the larger set until the two sets are equivalent. This action seems to be best modeled by the separating to (ST) and counting back to (CT) strategies while the matching strategy (M) is again appropriate for the comparison aspect of the problem.

Verbal Problems. For verbal problems, problem structure does appear to be the major determinant of solution strategy. (Table 4.)

Insert Table 4 about here

For the Separating problem almost three times as many subjects use a separating (S) or counting back (CB) strategy as used all the other strategies combined.

For the two Joining problems, the pattern of responses was almost identical. For each problem the adding on (AO) or counting up (CU) strategies were used almost twice as often as all the other strategies combined. With the Comparison problem, matching (M) was the dominant strategy.

The ambiguity of the Part-Part-Whole problem is reflected in the children's strategies which were about evenly divided between separating (S) and adding on (AO). Support for our analysis of the Equalizing problem is less strong, but it is generally consistent with the proposed model. Matching was a dominant strategy for both Equalizing problems, but in

both cases separating was used more frequently than the hypothesized separating to or adding on strategy. However, a comparison of the two Equalizing problems reveals that adding on (AO and CU) was used more frequently than separating to (ST and CT) for the addition problem (6 cases and 3 cases respectively) while the reverse was true for the subtraction problem (2 and 7 cases respectively).

Concrete Problems. For the concrete problems, the problem structure analysis does not predict performance nearly as well (Table 4). For four of the six problems, separating (S) was the principle strategy and for another separating (S) and separating to (ST) were employed with almost equal frequency. The only problem for which separating was not the dominant strategy was the second Joining problem. One explanation for this pattern of responses is that strategies were principly determined by the characterstics of the set of cubes subjects had available when they began to solve the problem. This is most clearly illustrated by the contrast between the strategies used to solve the two Joining problems. Although they both describe essentially the same action, they provide subjects with very different starting points. For example, consider the problem 11 - 3 = ?. In the first case the experimenter shows a subject a set of 3 cubes, adds some more cubes, and asks the subject to determine how many cubes were added. In the second example, a subject is given a set of 3 cubes and asked how many more are needed to have 11 cubes altogether. The key difference between the two problems is that in the first case the subject has 11 cubes to start with and can find the answer by simply removing 3 cubes. In the second case they must first construct the set of

11 cubes which is easiest to do by simply adding on to the set of 3. For every problem but the one joining problem, subjects had the larger set available and consequently relied primarily on a separating strategy.

Although problem structure was not the primary determinant of subjects' solution strategy it did appear to have some effect. The only use of the matching strategy (M) occured with the Comparison problem and one Equalizing problem, which is consistent with the analysis of problem structure. Comparing the two Equalizing problems reveals that 6 subjects used an adding on strategy (AO and CU) for the addition problem while none used it for the subtraction problem. On the other hand 16 subjects used the separating to strategy (ST and CT) with the subtraction problem while only 3 used it for the addition problem. Both of these results are consistent with the analysis of the two Equalizing problems.

One of the more interesting differences between the set of concrete problems and the set of verbal problems involves the use of the matching strategy. The matching strategy was used for every verbal problem at least twice and was a primary strategy for the Comparison problem and the two Equalizing problems. In the concrete set, however, it was only used on three problems for a total of nine times. It is not surprising that the matching strategy was not used for the Joining or Separating problems, where it would have been necessary to construct the second set. But for the Comparison and Equalizing problems both sets were already constructed. It is not clear why children would go to the trouble of constructing two sets to use a matching strategy in the verbal case and not use a matching

strategy in the concrete case, when the sets are already constructed. The matching strategy is actually more efficient for concrete problems than for verbal problems.

The most prevalent strategy overall was clearly the separating (S) strategy. Although the use of this strategy was not as overwhelming for verbal problems as with concrete problems, it was still the most commonly used strategy. It was the only strategy that was frequently used in contexts that were inconsistent with the analysis of problem structure. The choice of numbers in the subtraction problems (11 - 3 rather than 11 - 8) may have created some bias in favor of separating and counting back strategies, which may in part account for the popularity of the separating strategy. But no subjects indicated that number size influenced their choice of strategy. On the whole there is no basis for concluding that the choice of numbers had any influence on children's strategies. However, this is one limitation of this study, and additional research would be required to demonstrate conclusively that relative number size had no effect.

On the whole children were not quite as successful with the subtraction problems as they were with the addition problems. But over three-fourths of the subjects used the correct strategy, and well over half the responses were correct for every item. Furthermore, no one problem stood out as significantly more difficult than the others. Contrary to the findings of previous research with older children, very few children used the wrong operation. The most common error was to respond one of the given numbers but this accounted for at most six responses for any given problem.

Patterns of Children's Responses

This section focuses on the responses of individual children or groups of children over sets of related problems. The objective of this section is to attempt to identify groups of children who apply similar strategies and to characterize their pattern of responses.

One of the most interesting problems is to attempt to identify patterns of responses for individual children over groups of tasks so that it is possible to characterize a child's general strategy over the complete set of tasks. The different combinations of responses for the Joining and Part-Part-Whole addition problems are summarized in Table 5. Twenty-six used the same strategy for both verbal problems and 25 used

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Insert Table 5 about here

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the same strategy for both concrete problems. Thus although the two problems have very similar patterns of responses (Table 3) just over half the subjects used the same strategy for both problems.

It is a bit more difficult to identify general strategies for solving the subtraction problems because there are more problems and more distinct strategies that children can use for each problem. Most of the general strategies were defined in terms of the strategies used on individual problems. For example, a subject would be classified as using a general separating strategy if the subject almost always used a separating strategy.

A second major type of general strategy was based on problem structure. A subject was classified as using a problem structure strategy if he generally used one of the strategies that conform to the logical analysis of the problem structure. A subject was classified as using a particular strategy if the subject used the strategy for 5 of the 7 verbal problems or used it for 4 problems and used a number fact, heuristic, or uncodable strategy for the other problems. For the concrete problems the decision rule was 4 out of 6. Again the Part-Part-Whole problem was not included in the analysis.

Insert Table 6 about here

Over half of the subjects could be identified as using a particular general strategy (Table 6). The results are consistent with results for individual problems. The most frequently used general strategy for verbal problems was problem structure and for concrete problems it was separating.

In the analysis of individual subtraction problems several problems were identified that showed similar patterns of solution (Table 4). The two Joining missing addend problems (problems 5 and 6) had almost identical patterns of responses. However, an analysis of individual subject's responses reveals that although the overall pattern of responses were similar subjects were not especially consistent in responding to the two problems. Only 13 subjects used the same strategy for both problems. The two Equalizing problems and the Comparison problem also had somewhat similar patterns

of responses. For these three problems, 13 subjects used the same strategy on all 3 problems and an additional 18 used the same strategy on two out of three problems.

The use of the more sophisticated strategies is also of interest. Almost a third of the subjects used a heuristic strategy at least once, and almost three-fourths used at least one of the more advanced strategies (heuristic, counting up, or counting back).

Conclusions

A striking result of this study is the high level of success of first-grade children in solving verbal problems. Only four subjects used an incorrect strategy for more than half of the verbal problems, and over two-thirds used a correct strategy (one that would lead to a correct answer if applied accurately) for 8 of the 10 problems. Children were not only successful in modeling action or relationships implied in problems. They were also able to use different models of addition and subtraction when convenient and demonstrated some understanding of the inverse relationship between addition and subtraction. The fact that very few children relied exclusively on strategies that directly modeled the action in the problem further illustrates that their problem solving strategies involved some understanding of the nature of the operations.

The first-grade children in this study gave very little evidence of the types of systematic errors reported in previous studies. Very few of

them used the wrong operation in their solutions. Since this error has been observed primarily with older children who have already experienced formal instruction in addition and subtraction, it may actually be a result of learning symbolic representations. Typically addition and subtraction are introduced in terms of joining or separating sets using either pictures or concrete objects. Then children are drilled on abstract problems with number sentences. When they finally get to verbal problems, their response is, "Is this a plus or a takeaway?" Because the operations are initially learned outside of the context of verbal problems and children are simply told that addition and subtraction can be used to solve these problems, they have no basis for using their natural intuition to relate the problem structure to the operations they have learned. In other words, their natural analytic problem solving skills are bypassed, and they too often resort to relying on superficial problem characteristics to identify the correct operation. This may result not only in a superficial concept of addition and subtraction but also in a decline in general problem solving ability.

The results of this study suggest a somewhat different picture of children's processes for solving addition and subtraction problems than has been proposed in other analyses of these operations. Greeno (Note 2) hypothesizes that children associate solution strategies directly with the semantic content of problems rather than constructing sets of simultaneous equations based on syntactic information within the problem.

This analysis is consistent with the results of this study. However, Greeno also hypothesizes that some problems are associated directly with an operation while others are first transformed to different problem structures. Specifically joining missing addend problems and certain comparison problems are first transformed to part-part-whole problems.

The results of this study suggest a different hypothesis. The tremendous variability between and within children in the solution processes used suggest that before receiving formal instruction, young children do not transform problems into a single type and apply a single strategy. The results indicate that children have available a rich repertoire of strategies and that they make use of many of these to solve various problem types. It is still not clear what triggers the use of a particular strategy; but it seems plausible that children solve each problem type directly, rather than collapsing them and applying a single strategy consistently.

The picture painted by this description is quite different from that proposed by Greeno. In Greeno's description, the limiting factor is the number of different solution strategies which children have available. Since empirical data show that children can solve a variety of problem types it was assumed that they must transform them in order to successfully apply the few strategies which they have acquired. The results presented here suggest that even prior to instruction most children possess the different strategies which are required to solve each problem type directly. No transformations are needed. In fact, it may be the transformation process itself which is the limiting factor as children begin instruction.

Arithmetic instruction frequently illustrates a particular operation like subtraction with several problem types (eg., Separating, Part-Part-Whole, Comparison). Although most children can solve each type of problem using an appropriate strategy (e.g., separating, add on, matching), they may have trouble transforming these problems and understanding that a single strategy would be appropriate for all of them. This conjecture is supported by the small number of children in this study who used a single strategy consistently across problem types, and by the well documented difficulties which children experience with missing addend problems in most curriculum programs.

The results of this study also deviate to some degree from the results of earlier latency studies of children's solution of number sentences (Groen and Parkman, 1975; Groen and Resnick, 1977; Woods, Resnick and Groen, 1975). Specifically this study found less frequent use of counting on strategies for addition problems than was found in earlier studies (Groen and Parkman, 1975; Groen and Parkman, 1977); and although there was no direct test of the effect of number size, other factors seemed to have a greater influence in determining children's choice between adding on or counting back strategies. This study also identified two strategies, matching and heuristic, that were not even considered in the earlier studies.

To some extent these discrepencies may result from differences in the age of children in the sample and differences in characteristics of the problems. Certainly it is necessary to be very careful in making comparisons between the solution of verbal problems and the solution of number

sentences. Two factors that may contribute to the differences in performance are the different number domains and the availability of cubes. The larger numbers in this study perhaps make counting on or choice strategies less likely. The availability of cubes clearly seems to influence children to use a counting all rather than a counting on strategy. This is illustrated by the fact that the children who used the cubes almost always used a counting all strategy while those using fingers or no action generally used a counting on strategy. To some extent this may result from the fact that the more capable children, those most able to use more sophisticated counting on strategies, tended to be the ones who did not use cubes. But the cubes do appear to encourage children to model the complete problem.

One of the most fundamental differences between this study and the earlier studies is in the experimental paradigm: clinical interview as opposed to matching response latencies to predicted regression equations. The response latency paradigm is based on the assumption that children consistently apply a well defined strategy to their solution problems. The results of this study indicate that this assumption is at least suspect, and the results of response latency studies should be subjected to further validation. It also appears that one should be very careful in generalizing the results of this study or the response latency studies beyond the domain of problems included in the specific study.

The results of this study tend to support the hypothesis that verbal problems may be the most appropriate context in which to introduce addition and subtraction operations. Clearly, verbal problems are a viable alternative to traditional settings since children are able to interpret them

and generate solutions prior to formal instruction. Verbal problems also provide different interpretations of addition and subtraction, interpretations that are important for children to understand. Perhaps by introducing operations based on verbal problems and integrating verbal problems throughout the mathematics curriculum, rather than using them as an application of already learned algorithms, children will develop their natural ability to analyze problem structure and will develop a broader conception of basic operations.

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	Action	Static Relationships
Set Inclusion	Joining/Separating	Part-Part-Whole
No Set Inclusion	Equalizing	Comparison

Figure 1. Classes of Addition and Subtraction Problems.

TABLE 1

VERBAL PROBLEMS

Addition

1. Joining
   Wally had a pennies. His father gave him b more pennies. How many pennies did Wally have altogether?

2. Part-Part-Whole
   Some children were ice-skating. a were girls and b were boys. How many children were skating altogether?

3. Difference
   Ralph has a pieces of gum. Jeff has b more pieces than Ralph. How many pieces of gum does Jeff have?

Subtraction

4. Separating
   Leroy had a pieces of candy. He gave b pieces to Jenny. How many pieces of candy did he have left?

5. Joining (1)
   Susan had a books. Her teacher gave her some more books. Now she has c books altogether. How many books did Susan's teacher give her?

6. Joining (2)
   Kathy had a toys. How many more does she need to have c toys altogether?

7. Part-Part-Whole
   There are c children on the playground. a are boys and the rest are girls. How many girls are at the playground?

8. Difference
   Mark won a prizes at the fair. His sister Connie won c prizes. How many more prizes did Connie win than Mark?

9. Equalizing (+)
   Joan picked a flowers. Bill picked c flowers. What could Joan do so she could have as many flowers as Bill? (Suggest, if necessary, that she pick some more.) How many more would she need to pick?

10. Equalizing (-)
    Fred has a marbles. Betty has c marbles. What could Betty do so she would have as many marbles as Fred? (Suggest, if necessary giving some away.) How many would she need to get rid of?

TABLE 2

CONCRETE PROBLEMS

Addition

1. Joining
   Subject is asked to count separate sets of a red cubes and b white cubes. Cubes are physically combined and subject is asked how many cubes there are altogether.

2. Part-Part-Whole
   Subject is asked to count a red cubes and b white cubes in a mixed set. Subject is then asked how many cubes there are altogether.

3. Difference
   Subject is asked to count set of a red cubes. Subject is then asked to determine how many white cubes would be in a set which had b more white cubes than red cubes.

Subtraction

4. Separating
   Subject is asked to count set of a red cubes. Subject is then asked to determine how many cubes would be left if b cubes were removed.

5. Joining (1)
   Subject is asked to count set of a white cubes. A second set of white cubes is combined with the first to make one set of c white cubes. Subject is asked to determine how many cubes were added to the first set.

6. Joining (2)
   Subject is asked to count set of a white cubes. Subject is then asked to determine how many white cubes must be added to make a set of c white cubes.

7. Part-Part-Whole
   Subject is presented with sets of a white cubes and b red cubes and is asked to count total number of cubes and number of white cubes. Subject is then asked to determine number of red cubes.

8. Difference
   Subject is asked to count sets of a red cubes and c white cubes. Subject is then asked to determine how many more white cubes there are than red cubes.

TABLE 2 (Continued)

Subtraction (Cont.)

9. Equalizing (+)
   Subject is asked to count sets of a red cubes and c white cubes. Subject is asked to determine what must be done to the red set to make as many red cubes as white cubes. Subject is then asked to determine how many red cubes must be added to make the sets equal.

10. Equalizing (-)
    Subject is asked to count sets of a red cubes and c white cubes. Subject is asked to determine what must be done to the white set to make as many white cubes as red cubes. Subject is then asked to determine how many white cubes must be removed to make the sets equal.

TABLE 3

SUMMARY OF ADDITION RESPONSES

Problem	Model			Strategy						Total Correct		Error		
	C	F	N	CA	CF	CL	H	KF	U	V	A	M	CE	E
Verbal Problems														
Joining	23	7	13	21	4	4	5	1	6	41	34	1	6	2
Part-Part-Whole	24	6	13	22	3	9	2	1	1	38	37	1	0	5
Comparison	17	3	23	7	1	1	2	0	1	12	10	1	1	31
Concrete Problems														
Joining	33	1	9	26	1	7	5	2	2	43	40	1	2	0
Part-Part-Whole	32	4	7	26	2	9	3	1	0	41	37	1	3	2
Comparison	19	2	22	6	4	6	3	0	2	21	20	0	1	22

TABLE 4

SUMMARY OF SUBTRACTION RESPONSES

Problem	Model			Strategy										Total Correct		Errors			
	C	F	N	S	ST	AO	M	CB	CT	CU	H	KF	U	V	A	CE	G	O	E
Verbal																			
Separating	25	5	13	19	0	0	4	9	0	0	3	2	2	39	31	8	1	0	3
Joining 1	27	3	13	4	0	15	2	1	0	2	3	0	2	31	22	7	1	5	4
Joining 2	27	5	11	4	1	14	5	0	0	8	3	0	1	36	31	5	1	1	5
Part-Part-Whole	26	4	13	9	0	10	4	3	0	3	0	2	2	33	20	13	6	0	4
Comparison	31	2	10	8	1	2	17	0	0	3	3	1	0	35	29	6	3	0	5
Equalizing +	32	1	10	9	2	5	15	0	1	1	5	0	1	39	30	9	0	0	4
Equalizing -	32	3	8	14	4	0	13	1	3	2	2	0	0	39	29	10	0	0	4
Concrete																			
Separating	31	1	11	30	1	0	0	7	0	0	1	1	2	42	40	2	0	0	1
Joining 1	31	0	12	23	6	0	0	3	0	0	5	1	3	41	38	3	2	0	0
Joining 2	28	2	13	7*	0	16	0	0	0	4	2	1	3	33	31	2	1	0	9
Comparison	34	0	9	23	4	0	5	0	0	2	4	1	0	39	32	7	3	0	1
Equalizing +	31	2	10	18	3	4	2	2	0	2	2	1	1	35	28	7	1	0	7
Equalizing -	35	0	8	15	15	0	2	1	1	0	1	2	1	38	34	4	1	0	4

** The Part-Part-Whole problem was constructed faithfully according to problem structure. As a result subjects were able to count the cubes in the unknown part which virtually all of them did. Consequently this problem was dropped from further analysis.

* Includes 5 responses that subjects started to add on and then switched to separating strategy.

TABLE 5

RESPONSES FOR JOINING AND PART-PART-WHOLE ADDITION PROBLEMS

Response Combinations	Verbal	Concrete
Both counting all	17	20
Both counting on	6	3
Both heuristic	3	2
Counting all-counting on	4	8
Heuristic-counting on	3	4
Heuristic-counting all	0	2
Other	10	4

TABLE 6

CLASSIFICATION OF GENERAL SUBTRACTION STRATEGIES

Strategy	Verbal	Concrete
Single strategy		
Separating	3	13
Matching	4	0
Heuristic	2	3
Structure of Problem	14	6
Consistent Error	3	3
Unclassifiable	17*	18

*Includes 8 subjects who used only two strategies 4 of whom used AO and S.